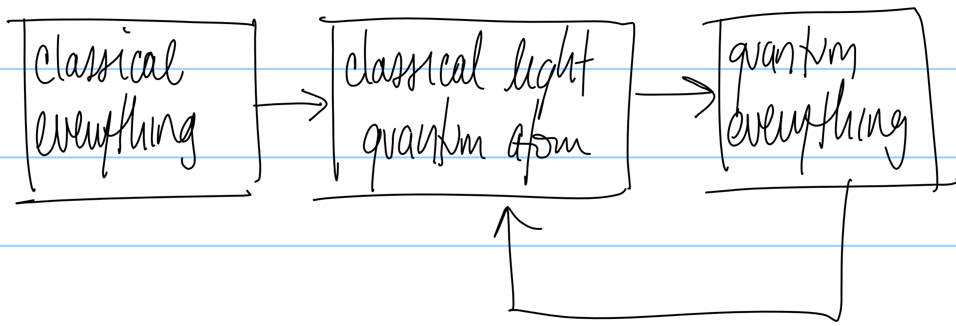
PHYS598MAP

Note Title 3/16/2005

Quantization of the electromagnetic field

Almost the final step in our roadmap:



How do we make the light-field quantum?

QED

But we're not that smart, so we'll quantize the electromagnetic field in a standard, pretty-good AMO way. The story goes like this...

What do we care about? Light interacting with an atom! In our Hamiltonian, we have to replace

$\vec{p} \rightarrow \vec{p} + e\vec{A}$ (I will try to get mks units right, but a lot of people use c.g.s.)

We already know how to deal with $\vec{p}$ as an operator, but we have to figure out what to do with $\vec{A}$.

Plan: ① 1st quantization of $\vec{A}$: write down purely classical expansion of $\vec{A}$ in terms of normal modes

② 2nd quantization of $\vec{A}$: associate normal modes with a QM eigenbasis (harmonic oscillator)

① Classical E&M:

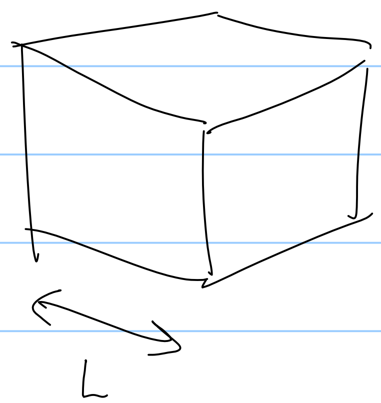
use Coulomb gauge: $\vec{\nabla}\cdot\vec{A}=0$

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}\qquad \vec{B}=\vec{\nabla}\times\vec{A}$$

(in source-free regions, the fields are completely transverse)

We end up with a wave equation: $\nabla^2 A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}$

And solve it using a fictitious box:



Note: our final answer for observables better not depend on anything about this box!

$$\vec{A} = \sum_{\lambda j}\left\{ q_\lambda e^{-i\omega_\lambda t}\vec{A}_{\lambda j} + q_\lambda^* e^{i\omega_\lambda t}\vec{A}_{\lambda j}^{\,*}\right\}$$

λ is a "mode": $\vec{A}_{\lambda j} = \frac{1}{\sqrt{2\varepsilon_0 V}} e^{i\vec{k}_\lambda\cdot\vec{r}}\hat{\varepsilon}_{\lambda j}$

$$\vec{k}_\lambda = \frac{2\pi}{L}\left(\nu_x\hat{x} + \nu_y\hat{y} + \nu_z\hat{z}\right) \quad \nu_i = 0, \pm 1, \pm 2, \ldots$$

$\frac{1}{\sqrt{V}}$ is a normalization constant

waves are transverse: $\hat{\varepsilon}_{\lambda j}\cdot\vec{k}_\lambda = 0$

2 polarizations! $\hat{\varepsilon}_{\lambda j}\cdot\hat{\varepsilon}_{\lambda j'} = \delta_{jj'}$

Plane wave expansion useful for free space, but other choices might be more convenient

(like cavity modes)

that was 1st quantization!

the q_λ's keep track of how much field amplitude is in each mode:

$$\vec{E} = i \sum_{\lambda j} \left\{ \omega_\lambda q_\lambda e^{-i\omega_\lambda t} \vec{A}_{\lambda j} - \omega_\lambda q_\lambda^* e^{i\omega_\lambda t} \vec{A}_{\lambda j}^* \right\}$$

$$\vec{B} = i \sum_{\lambda j} \left\{ q_\lambda e^{-i\omega_\lambda t} \vec{k}_\lambda \times \vec{A}_{\lambda j} - q_\lambda^* e^{i\omega_\lambda t} \vec{k}_\lambda \times \vec{A}_{\lambda j}^* \right\}$$

② What's the energy?

$$H_{em} = \frac{1}{2} \int_V d^3x \left\{ \varepsilon_0 |E|^2 + \frac{1}{\mu_0} |B|^2 \right\}$$

⋮ math
⋮

$$H_{em} = \frac{1}{2}\sum_\lambda \omega_\lambda^2 q_\lambda q_\lambda^*$$ this looks suspiciously familiar!

introduce "position" and "momentum" coordinates (which are real valued):

$$Q_\lambda = \frac{1}{\sqrt{2}}(q_\lambda + q_\lambda^*)$$

$$P_\lambda = \frac{i\omega_\lambda}{\sqrt{2}}(q_\lambda^* - q_\lambda)$$

Then:

$$H_{em} = \tfrac{1}{2}\sum_\lambda (P_\lambda^2 + \omega_\lambda^2 Q_\lambda^2)$$

This is a classical harmonic oscillator! $(m=1)$

What are the coordinates? More soon...

We know what to do with this sucker to make it quantum...

$$a_\lambda^\dagger = \frac{1}{\sqrt{2\hbar\omega_\lambda}}\left(\omega_\lambda \hat{Q}_\lambda - i\hat{P}_\lambda\right)$$

$$a_\lambda = \frac{1}{\sqrt{2\hbar\omega_\lambda}}\left(\omega_\lambda \hat{Q}_\lambda + i\hat{P}_\lambda\right)$$

$$[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}$$

$$[a_\lambda, a_{\lambda'}] = [a_\lambda^\dagger, a_{\lambda'}^\dagger] = 0$$

<u>Then</u>: $$\boxed{H_{em} = \sum_\lambda \hbar\omega_\lambda\left(a_\lambda^\dagger a_\lambda + 1/2\right)}$$

or

$$\boxed{H_{em} = \sum_\lambda \hbar\omega_\lambda\left(n_\lambda + 1/2\right)} \qquad n_\lambda = a_\lambda^\dagger a_\lambda$$

<u>Note</u>: H_{em} is diagonal in the photon-number representation:

$$a_\lambda^\dagger |\ldots n_\lambda \ldots\rangle = \sqrt{n_\lambda + 1}\,|\ldots (n_\lambda+1)\ldots\rangle$$

$$a_\lambda |\ldots n_\lambda \ldots\rangle = \sqrt{n_\lambda}\,|\ldots (n_\lambda-1)\ldots\rangle$$

n_λ = # "photons" in mode λ

$$H_{em} | 0 \ldots n_\lambda \ldots 0 \rangle = \hbar\omega_\lambda (n_\lambda + 1/2) | 0 \ldots n_\lambda \ldots 0 \rangle$$

energy of each mode quantized in units of ω_λ

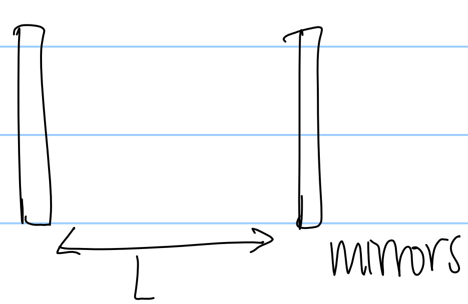
<u>total energy:</u> $\langle n_1 \ldots n_\lambda \ldots | H_{em} | n_1 \ldots n_\lambda \ldots \rangle =$

$$= \underbrace{\sum_\lambda \hbar\omega_\lambda n_\lambda}_{n_\lambda \text{ photons/mode}} + 1/2 \underbrace{\sum_\lambda \hbar\omega_\lambda}_{= \infty! \text{ vacuum energy}}$$

<u>What's up with the infinite vacuum energy?</u>

① We don't know
② Hard to see, but seems real

<u>Casimir effect</u> –



in 1-D: $E = \sum \hbar c k = \frac{\pi \hbar c}{L} \sum_{\nu \geq 1}^{\infty} \nu$ vacuum energy

without the cavity: $E = \frac{\pi \hbar c}{L} \int_0^{\infty} d\nu \, \nu$

$$\Delta E = \frac{\pi \hbar c}{L} \underbrace{\left(\sum_{\nu=1}^{\infty} \nu - \int_0^{\infty} \nu \, d\nu \right)}_{= -\frac{1}{12}}$$ shift because of the presence of the cavity

regularized by introducing convergence factor $e^{-\varepsilon \nu}$ and taking $\lim_{\varepsilon \to 0}$

$\Delta E = \frac{-\pi \hbar c}{12 L} \longrightarrow F = \frac{-\partial(\Delta E)}{\partial L} = \frac{-\pi \hbar c}{12 L^2}$

The Casimir force!

→ experiments have seen this (with conductors and lenses, or between BECs and surfaces)

Usually — shift our zero for Ham and ignore the vacuum energy

Remember our friend $\vec{A}$?

$$\vec{A} = \sum_{\lambda j} \left\{ q_\lambda e^{-i\omega_\lambda t} \vec{A}_{\lambda j} + q_\lambda^* e^{i\omega_\lambda t} \vec{A}_{\lambda j}^* \right\}$$

$$\left\{ \begin{array}{l} \hat{Q}_\lambda = \sqrt{\frac{\hbar}{2\omega_\lambda}} \left(a_\lambda^\dagger + a_\lambda \right) \\ Q_\lambda = \frac{1}{\sqrt{2}} \left(q_\lambda + q_\lambda^* \right) \end{array} \right.$$

$$\left\{ \begin{array}{l} q_\lambda \to \sqrt{\frac{\hbar}{\omega_\lambda}} a_\lambda \\ q_\lambda^* \to \sqrt{\frac{\hbar}{\omega_\lambda}} a_\lambda^\dagger \end{array} \right.$$

$$\boxed{\vec{A} = \sum_{\lambda j} \sqrt{\frac{\hbar}{\omega_\lambda}} \left(a_{\lambda j} e^{-i\omega_\lambda t} \vec{A}_{\lambda j} + a_{\lambda j}^\dagger e^{i\omega_\lambda t} \vec{A}_{\lambda j}^* \right)}$$

We did the 2^{nd} quantization!

Note: $\vec{A}$ is not diagonal in photon number (if it's in a Hamiltonian, then n_λ is not constant!)

Note: our "photons" are non-local objects!

the photon is: $e^{i\omega_j t}\vec{A}^*_{\lambda j} = e^{i\omega_j t}\dfrac{1}{\sqrt{2\varepsilon_0 V}}e^{-i\vec{k}_j\cdot\vec{r}}\hat{\varepsilon}^*_{\lambda j}$

it's spread out over space!
it's not a wavepacket

Can we write down a photon wavefunction?

This is an open and disputed subject!
One answer: no

$\vec{r}$ is not an operator in our theory - the only "coordinates" in the Hamiltonian are $\hat{q}$ and $\hat{p}$

This situation is fraught with problems - see Scully and Zubairy, Quantum Optics.

Quantum States of Light

Note: What are those operators/coordinates $\hat{Q}$ and $\hat{P}$?

check it: $\quad \chi_\lambda = \omega_\lambda t - k_\lambda \cdot \vec{r} - \pi/2 \quad$ phase angle

write $\vec{E} = \vec{E}_+ + \vec{E}_-$:

$$\begin{cases} \vec{E}_+ = \sum_{\lambda j} \hat{\varepsilon}_{\lambda j} \sqrt{\frac{\hbar\omega_\lambda}{2\varepsilon_0 V}}\, a_{\lambda j}\, e^{-i\chi_\lambda(\vec{r},t)} \\ \vec{E}_- = \sum_{\lambda j} \hat{\varepsilon}_{\lambda j} \sqrt{\frac{\hbar\omega_\lambda}{2\varepsilon_0 V}}\, a^\dagger_{\lambda j}\, e^{i\chi_\lambda(\vec{r},t)} \end{cases}$$

and similar expressions for $\vec{B}_\pm$

$$\left.\begin{array}{l} \hat{Q}_\lambda = \sqrt{\frac{\hbar}{2\omega_\lambda}}\,(a^\dagger_\lambda + a_\lambda) \\ \hat{P}_\lambda = i\sqrt{\frac{\hbar\omega_\lambda}{2}}\,(a^\dagger_\lambda - a_\lambda) \end{array}\right\} \longrightarrow \begin{array}{l} \hat{X} = \frac{1}{2}(a^\dagger_\lambda + a_\lambda) = \sqrt{\frac{\omega_\lambda}{2\hbar}}\,\hat{Q}_\lambda \\ \hat{Y} = \frac{1}{2} i (a^\dagger_\lambda - a_\lambda) = \sqrt{\frac{1}{2\hbar\omega_\lambda}}\,\hat{P}_\lambda \end{array}$$

$\hat{X}$ and $\hat{Y}$ are rescaled versions of our real-valued coordinates called quadrature operators

use: $a = X + iY$, $a^{\dagger} = X - iY$ $\Big\}$ $\rightarrow$ plug into equation for $\vec{E}$

$$\vec{E} = \sum_{\lambda j} \sqrt{\frac{2\hbar\omega_\lambda}{\varepsilon_0 V}}\, \hat{\varepsilon}_{\lambda j} \left\{ X_{\lambda j} \cos\left[\chi_\lambda(\vec{r}, t)\right] + Y_{\lambda j} \sin\left[\chi_\lambda(\vec{r}, t)\right] \right\}$$

these operators will be useful for creating graphical representations of the real-valued field

Example: $|n\rangle$ states (pick one mode) $= \boxed{\text{"Fock states"}}$

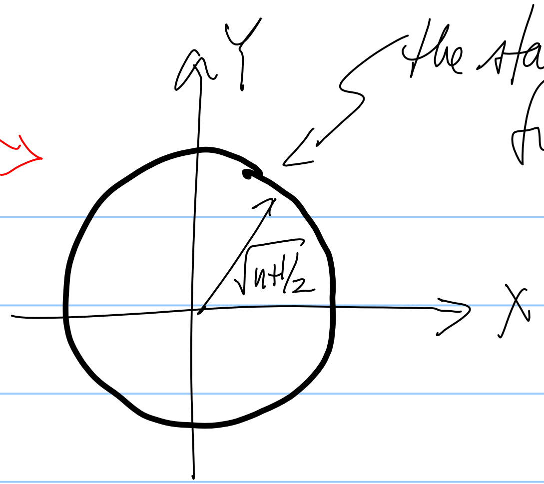
$$H_{em} |n\rangle = \hbar\omega \left(X^2 + Y^2\right) |n\rangle = \hbar\omega \left(n + \tfrac{1}{2}\right) |n\rangle$$

$$\left(X^2 + Y^2\right) |n\rangle = \left(n + \tfrac{1}{2}\right) |n\rangle$$

$$\langle n | X | n \rangle = \langle n | Y | n \rangle = 0$$

$$(\Delta X)^2 = (\Delta Y)^2 = \tfrac{1}{2}\left(n + \tfrac{1}{2}\right)$$

all plots like this have E in units of $\sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}}$

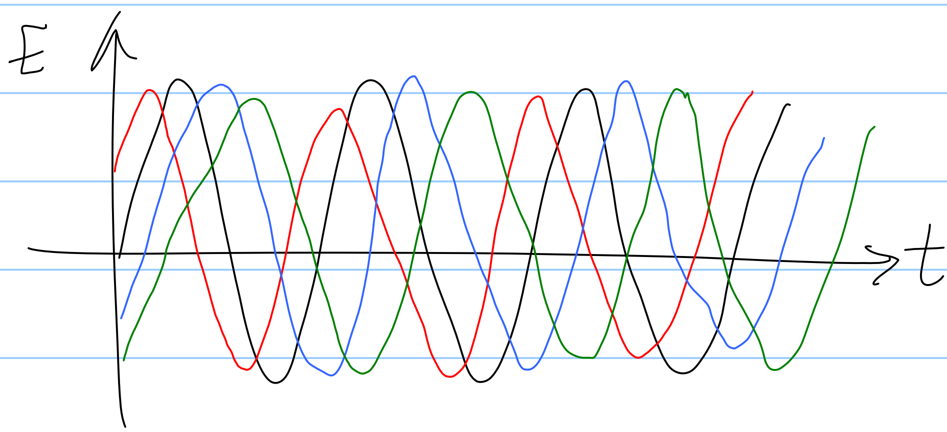


the state of the electric field lies on this line, but the average value is 0

$$\langle n|E|n\rangle = 0$$

$$(\Delta E)^2 = \langle n|(E)^2|n\rangle = \tfrac{1}{2}(n+\tfrac{1}{2})\sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}}$$

People think of this like:



→ each photon in the mode has the same field strength, but a random phase angle

A Fock state is a quantum state of light. Another quantum state of light is a coherent state.

Coherent states

Special superposition of Fock states that "looks like" a 'classical' state.

Where do they come from?

Consider: classical current $\vec{J}$ interacting with the quantized EM field

interaction Hamiltonian: $H_I = \int \vec{J} \cdot \vec{A} \, d^3r$

not an operator

operator

what is the state of the field?

In the interaction picture: $\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} H_I |\psi\rangle$

state of the field in 2nd quantized notation

Math... math... math... integration...

$$e^{-i/\hbar \int_0^t dt' H_I(t')} \Rightarrow \prod_\lambda e^{\alpha_\lambda a_\lambda - \alpha_\lambda^* a_\lambda}$$

(up to an overall phase)

<u>where:</u> α_λ = c-number =

$$\frac{i}{\sqrt{\hbar\omega_\lambda}} \frac{1}{\sqrt{2\varepsilon_0 V}} \int_0^t dt' \int d\vec{r}\, \hat{\varepsilon}_j \cdot \vec{J}\, e^{i\omega_\lambda t - i\vec{k}_\lambda \cdot \vec{r}}$$

If $|\psi(t=0)\rangle = |0\rangle$, then

$$|\psi(t)\rangle = \prod_\lambda e^{\alpha_\lambda a_\lambda^\dagger - \alpha_\lambda^* a_\lambda} |0_\lambda\rangle$$

This is called a coherent state!

$$\boxed{|\alpha_\lambda\rangle = \underbrace{e^{\alpha_\lambda a_\lambda^\dagger - \alpha_\lambda^* a_\lambda}}_{\text{displacement operator } D(\alpha)} |0_\lambda\rangle}$$

What is this state in the number-state basis?

consider a single mode:

$$|\alpha\rangle = D(\alpha)|0\rangle$$

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

using: $e^{A+B} = e^{-[A,B]/2} e^A e^B$ (Baker-Hausdorff formula — need $[[A,B],A] = [[A,B],B] = 0$

$$D(\alpha) \rightarrow e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{\alpha^* a}$$

and $e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{\alpha^* a}|0\rangle =$

$$e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle \qquad (\text{since } e^{\alpha^* a}|0\rangle = |0\rangle)$$

using: $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle}$$

A few remarks about $D(\alpha)$:

$D^\dagger(\alpha) D(\alpha) = 1$ unitary

$D(\alpha)$ is like a creation operator for coherent states

adding a forcing term to a quantum harmonic oscillator creates a coherent state

displacement operator will "displace" a coherent state:

*note: $a|\alpha\rangle = \alpha|\alpha\rangle$ alternative definition of coherent state

what's $a[D(\beta)|\alpha\rangle]$?

$= D(\beta) \underbrace{D^\dagger(\beta)\, a\, D(\beta)}_{\text{handy trick}} |\alpha\rangle$

$= D(\beta)(a+\beta)|\alpha\rangle$

$$\rightarrow (\alpha+\beta)\,[D(\beta)\,|\alpha\rangle]$$

so: $D(\beta)|\alpha\rangle \longrightarrow |\alpha+\beta\rangle$

Coherent states:

overcomplete: $\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = 1$

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$$

Poisson distribution in n: $P_n = |\langle n|\alpha\rangle|^2 = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}$

for $n \gg 1$: $P_n = \frac{1}{\sqrt{2\pi\langle n\rangle}}\, e^{-\frac{(n-\langle n\rangle)^2}{2\langle n\rangle}}$

(Gaussian distribution)

mean # photons: $\langle n\rangle = \langle\alpha|n|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$

variance: $(\Delta n)^2 = |\alpha|^2$

$$\boxed{\frac{\Delta n}{\langle n\rangle} = \frac{1}{|\alpha|} = \frac{1}{\sqrt{\langle n\rangle}}}$$

To make a picture of this state:

$$\langle\alpha|X|\alpha\rangle = \frac{1}{2}\langle\alpha|(a^\dagger + a)|\alpha\rangle = \tfrac{1}{2}(\alpha^* + \alpha) = \mathrm{Re}(\alpha)$$

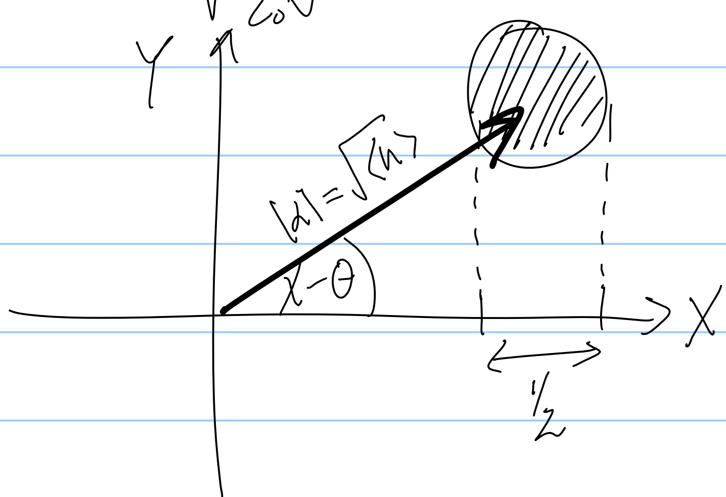
$$= |\alpha|\cos\theta \quad \left(\text{where } \alpha = |\alpha| e^{i\theta}\right)$$

$$\langle\alpha|Y|\alpha\rangle = |\alpha|\sin\theta$$

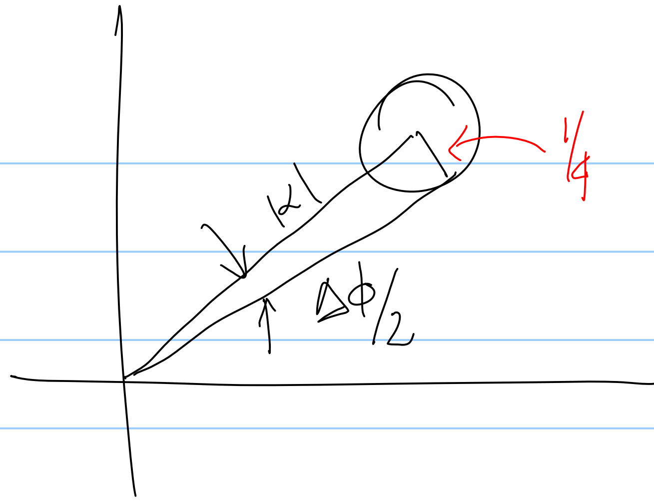
$$(\Delta X)^2 = (\Delta Y)^2 = 1/4$$

$$\langle\alpha|\hat{E}|\alpha\rangle = |\alpha|\cos(\chi-\theta)\sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}}$$

$$(\Delta E)^2 = 1/4\sqrt{\frac{2\hbar\omega}{\varepsilon_0 V}}$$



People often talk about a number-phase "uncertainty relation" (which is just geometry, and has nothing to do with conjugate variables)



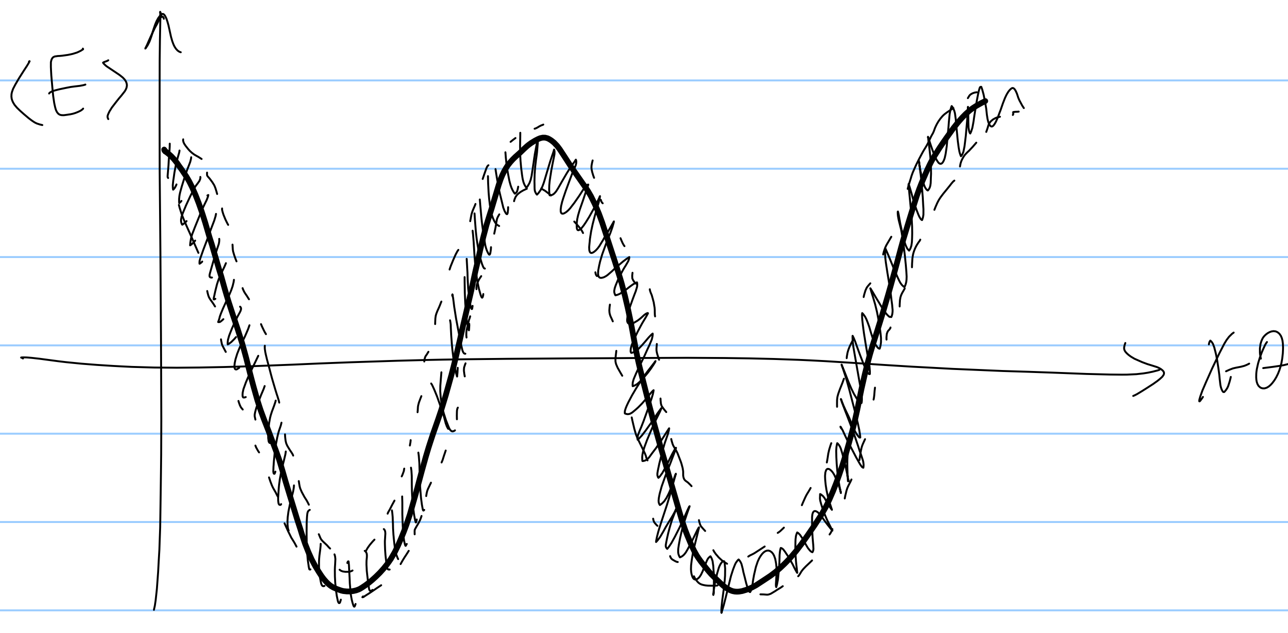
$$\sin\frac{\Delta\phi}{2} = \frac{1}{4|\alpha|}$$

$$\frac{\Delta\phi}{2} \sim \frac{1}{4|\alpha|}$$

$$\Delta\phi \sim \frac{1}{2|\alpha|}$$

$$= \frac{1}{2\sqrt{n}}$$

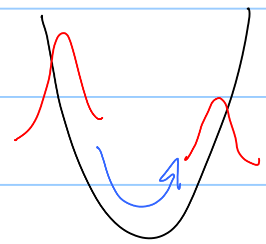
So, $\Delta n \Delta\phi \simeq 1/2$



So, the field "looks" classical, except for some quantum "fuzz" or "phase uncertainty"

What is so "coherent" about a coherent state?

two answers: ① for a quantum harmonic oscillator:

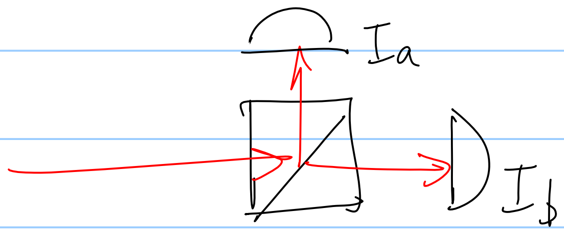


a coherent state is a wavepacket that oscillates ($\langle x \rangle \propto \cos(\omega t)$) without changing shape ($(\Delta x)^2 = \text{constant}$)

② $g^{(2)}(\tau) = 1$ 2nd order correlation

$$g^{(2)}(x_1; x_2) = \frac{\langle E_-(x_1) E_-(x_2) E_+(x_2) E_+(x_1) \rangle}{\langle E_-(x_1) E_+(x_1) \rangle \langle E_-(x_2) E_+(x_2) \rangle}$$

equivalent to Brown-Twiss interferometer:



measure intensities (clicks) in these detectors

$$g^{(2)}(\tau) = \frac{\langle a^\dagger(0) b^\dagger(\tau) b(\tau) a(0) \rangle}{\langle a^\dagger(0) a(0)\rangle \langle b^\dagger(\tau) b(\tau)\rangle}$$ quantum version

if a and b are orthogonal "modes"

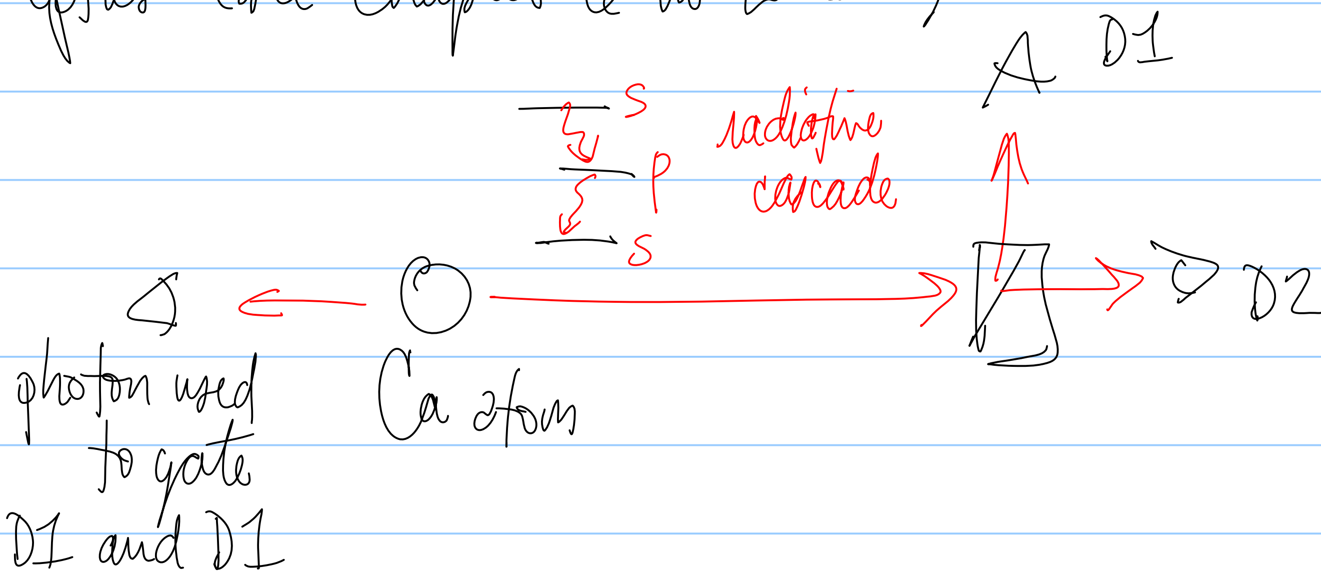
$$g^{(2)}(\tau) = \frac{\langle I_a(0) I_b(\tau)\rangle}{\langle I_a(0)\rangle\langle I_b(\tau)\rangle}$$

for a coherent state $g^{(2)}(\tau)$ is always 1, since the distribution in $|n\rangle$ is Poisson-ian (detection events are independent)

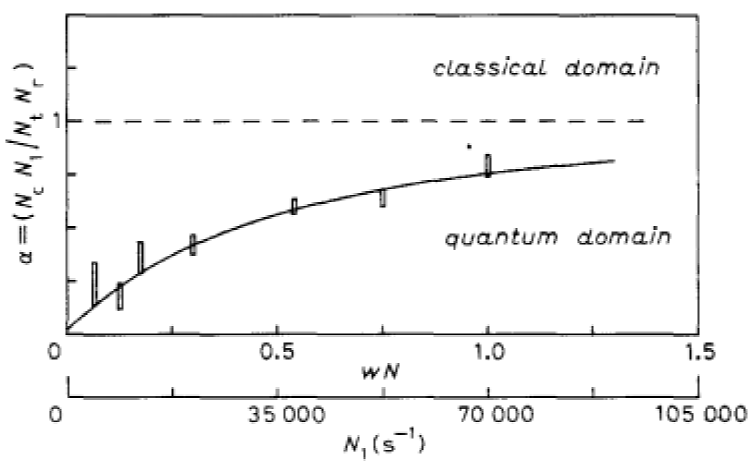
- compare with a Fock state: $g^{(2)}(\tau) = 1 - \frac{1}{\langle n \rangle}$

once you measure the "photon", then it's gone!

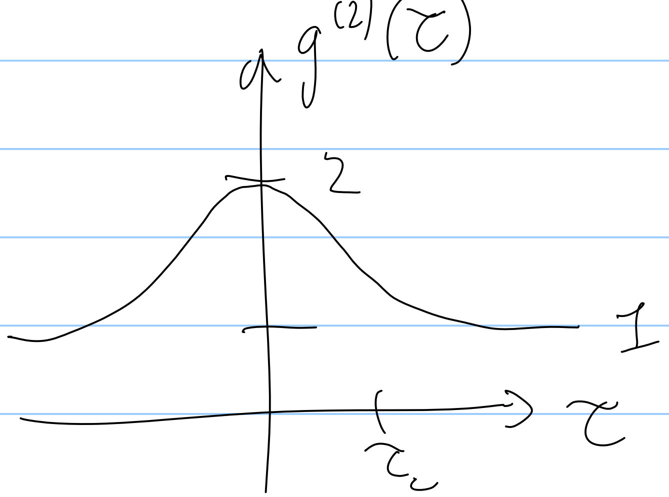
Something like this was measured by Aspect (see Europhys Lett. 1, 173 (1986)). To understand this experiment rigorously, you need to look at "continuous mode quantum optics" (see Chapter 6 in Loudon).



Basically $g^{(2)}(\tau)$ $\rightarrow$



If you do this experiment with a gas lamp or a light bulb, then you find:



τ_c: "coherence time" — time for relative phase of electric dipole for different emitters to randomize (basically the collision time in a lamp)

For a classical, "stable wave" $E = E_0 e^{ikz - i\omega t + i\varphi}$, $g^{(2)}(\tau) = e^{-i\omega_0 \tau}$, or $|g^{(2)}(\tau)| = 1$. We say such a source is "second order coherent."

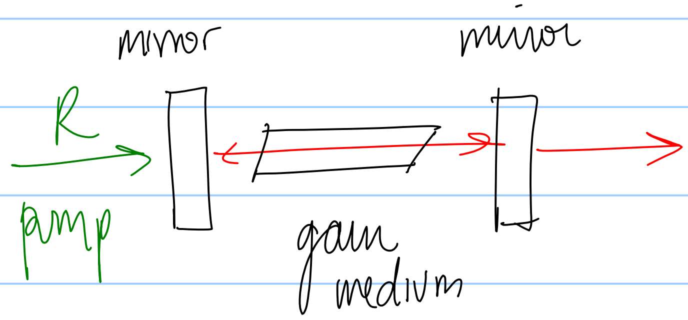
Other quantum states of light

Squeezed states
Phase states ...

Incidentally, what is the quantum state of light inside of a laser?

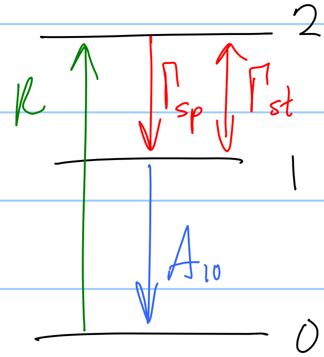
Many great references on this topic

Simple model from Loudon:



single-mode laser

gain medium: collection of atoms that are 3-level systems



Γ_{sp}: "spontaneous emission rate"

Γ_{st}: "stimulated emission rate" simple model using rate equations in the limit $A_{10} \gg R, \Gamma_{sp}, \Gamma_{st}$:

threshold: gain wins over losses

all other things equal: $C \propto R$ (cooperativity parameter)

n_s: saturation photon number

above threshold: $$P_n \simeq \frac{1}{\sqrt{2\pi(n_s+\langle n\rangle)}} e^{-\frac{(n-\langle n\rangle)^2}{2(n_s+\langle n\rangle)}}$$

like a coherent state, but with increased variance: $(\Delta n)^2 = n_s + \langle n\rangle$

The story of what happens inside of a laser is very complicated

- <u>note</u>: most people believe that the light which comes out of a laser is in a coherent state (or that it becomes a coherent state very quickly after bouncing off of a mirror or two)